Meromorphic differentials, Riemann-Roch theorem and Dimensions of the spaces of modular forms.

Projective line

- As a set:

$$\mathbb{P}^1(\mathbb{C}) := \mathbb{C} \amalg \{\infty\}$$

- As topological space:

For $R > 0$ $\quad \mathcal{U}_R := \{ z \in \mathbb{C} \mid |z| > R \}$

$\mathcal{U}_R$ is a neighbourhood of ∞ at $\mathbb{P}^1(\mathbb{C})$

- As a Riemann surface:

We define two holomorphic charts

$\Omega_1 := \mathbb{C} \qquad \varphi_1 : \Omega_1 \to \mathbb{C} \qquad z \mapsto z$

$\Omega_2 := \{\infty\} \cup \mathbb{C} \setminus \{0\} \qquad \varphi_2 : \Omega_2 \to \mathbb{C} \qquad z \mapsto \begin{cases} 0, & \text{if } z = \infty \\ \frac{1}{z}, & \text{if } z \in \mathbb{C} \setminus \{0\} \end{cases}$

Transition map: $\mathbb{C} \setminus \{0\} \xrightarrow{\varphi_1^{-1}} \Omega_1 \cap \Omega_2 \xrightarrow{\varphi_2} \mathbb{C} \setminus \{0\} \qquad \varphi_2 \circ \varphi_1^{-1}(z) = \frac{1}{z}$

Meromorphic functions

Definition: Let X be a Riemann surface. A **meromorphic function** on X is a holomorphic function from X to $\mathbb{P}^1(\mathbb{C})$.

Notation: The set of meromorphic functions on X is denoted by $\mathbb{C}(X)$.

Proposition: $\mathbb{C}(X)$ is a field with respect to pointwise addition and multiplication.

Example: Meromorphic functions on modular curves.

Let Γ be a congruence subgroup of $SL_2(\mathbb{Z}) =: \Gamma_1$.

Let $f : \mathfrak{h} \to \mathbb{C}$ be a meromorphic function.

Suppose that:

- $f\left(\frac{az+b}{cz+d}\right) = f(z)$ for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ and $z \in \mathfrak{h}$.
- Around each cusp $\alpha = \frac{a_1 \infty + b_1}{c_1 \infty + d_1}$, $\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \in \Gamma_1$, f has the Laurant expansion:

$$f\left(\frac{a_1 z + b_1}{c_1 z + d_1}\right) = \sum_{n=n_0}^{\infty} c_{f,\alpha}(n)\, e^{\frac{2\pi i}{h_\alpha} \cdot n z}$$

for some $h_\alpha \in \mathbb{Z}_{>0}$ and $n_0 \in \mathbb{Z}$

Then f defines a meromorphic function on $X(\Gamma)$.

Example: $\mathbb{C}(X_1)$

The **elliptic j-invariant** is defined as

$$j(z) := \frac{1728\, E_4(z)^3}{E_4^3(z) - E_6^2(z)}, \quad z \in \mathfrak{h}$$

Expansion around the ∞:

$$j(z) = e^{-2\pi i z} + 744 + 196884\, e^{2\pi i z} + 21493760\, e^{4\pi i z} + \dots$$

$j : X_1 \to \mathbb{P}^1(\mathbb{C})$ is a meromorphic function.

Exercise: show that

- $j : X_1 \to \mathbb{P}^1(\mathbb{C})$ is a biholomorphic map
- $\mathbb{C}(X_1) = \mathbb{C}(j)$ (any meromorphic function on X_1 is a rational function of j)

Non-zero weight.

Let $k \in \mathbb{Z} \setminus \{0\}$, Γ be a congruence subgroup.

Suppose that $f : \mathfrak{h} \to \mathbb{P}^1(\mathbb{C})$ is a meromorphic function and

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z) \text{ for all}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma, \quad z \in \mathfrak{h}.$$

Then f is not well-defined on $X(\Gamma)$.

Meromorphic differentials.

Definition: Let $\mathcal{U} \subset \mathbb{C}$ be an open subset.

A **meromorphic differential** (of degree 1) on $\mathcal{U}$ is a symbol of the form

$$f(u)\,du$$

where $f : \mathcal{U} \to \mathbb{P}^1$ is a meromorphic function and u is a variable on $\mathcal{U}$.

- Change of the variable $\quad u, v$ - two variables on $\mathcal{U}$

$$f(u)\,du = f(u(v)) \cdot \frac{d}{dv}u(v) \cdot dv$$

Notation: $\Omega^1(\mathcal{U}) := \{ f(u)\,du \}$ the space of differential forms on $\mathcal{U}$.

Operations with meromorphic differentials:

- Addition $\underbrace{f_1(u)\,du}_{\text{meromorphic differential}} + \underbrace{f_2(u)\,du}_{\text{meromorphic differential}} = (f_1+f_2)(u)\,du$

- Multiplication by a function $\underset{\substack{\uparrow \\ \text{meromorphic function}}}{f_1} \cdot \left(f_2(u)\,du \right) = \left(f_1 \cdot f_2\right)(u)\,du$

Meromorphic differentials of higher degree

Definition: Let $\mathcal{U} \subset \mathbb{C}$ be an open subset. A **meromorphic differential of degree n** is a symbol of the form

$$f(u)(du)^n$$

where $f : \mathcal{U} \to \mathbb{P}^1$ is a meromorphic function and u is a variable on $\mathcal{U}$.

- Change of the variable

u, v - two variables on $\mathcal{U}$

$$f(u)(du)^n = f(u(v)) \cdot \left(\frac{d}{dv} u(v)\right)^n \cdot (dv)^n$$

Notation: $\Omega^{\otimes n}(\mathcal{U}) := \{ f(u)(du)^n \}$ the space of differential forms of degree n on $\mathcal{U}$.

Operations with meromorphic differentials

- Addition $f_1(u)(du)^n + f_2(u)(du)^n = (f_1+f_2)(u)(du)^n$

- Multiplication by a function $f_1 \cdot (f_2(u)(du)^n) = (f_1 \cdot f_2)(u)(du)^n$

- Multiplication of differentials $\left(f_1(u)(du)^n\right) \cdot \left(f_2(u)(du)^m\right) =$

$$= (f_1 \cdot f_2)(u)(du)^{n+m}$$

Pullback

Let $\mathcal{U}_1, \mathcal{U}_2 \subset \mathbb{C}$ be open subsets, respectively, with local coordinates u_1, u_2, respectively. Let $\varphi : \mathcal{U}_1 \to \mathcal{U}_2$ be a holomorphic map.

The pullback map

$$\varphi^* : \Omega^{\otimes n}(\mathcal{U}_2) \longrightarrow \Omega^{\otimes n}(\mathcal{U}_1)$$

is defined by

$$\varphi^*\left(f(u_2)(du_2)^n \right) = f(\varphi(u_1))\left(\frac{d\varphi}{du_1}\right)^n \cdot (du_1)^n.$$

Properties of the pullback map

- $V_1 \xrightarrow{\varphi_1} V_2 \xrightarrow{\varphi_2} V_3$

  $$\begin{tikzcd} \Omega^{\otimes n}(V_3) \arrow[r, "\varphi_2^*"] \arrow[rr, bend right, "(\varphi_2 \circ \varphi_1)^*"'] & \Omega^{\otimes n}(V_2) \arrow[r, "\varphi_1^*"] & \Omega^{\otimes n}(V_1) \end{tikzcd}$$

  $$(\varphi_2 \circ \varphi_1)^* = \varphi_1^* \circ \varphi_2^*$$

- $i : V_1 \to V_2$ is inclusion, then
  $i^* : \Omega^{\otimes n}(V_2) \to \Omega^{\otimes n}(V_1)$ is restriction

- $\varphi : V_1 \to V_2$ is a bijection (then also a bi-holomorphism)
  then $(\varphi^{-1})^* = (\varphi^*)^{-1}$

- If $\pi : V_1 \to V_2$ is a surjection then π^* is injective.

Meromorphic differentials on a Riemann surface

Let X be a Riemann surface

Let $\varphi_j : U_j \to V_j$, $j \in J$ be an atlas of holomorphic charts on X.

A **meromorphic differential of degree n** on X is a collection of meromorphic differentials

$$\omega_j \in \Omega^{\otimes n}(V_j), \quad j \in J$$

compatible with pullbacks by transition maps:

$$\psi_{i,j} = \varphi_i \circ \varphi_j^{-1} : \underbrace{\varphi_j(U_i \cap U_j)}_{\subseteq V_j} \to \underbrace{\varphi_i(U_i \cap U_j)}_{\subseteq V_i}$$

$$\psi_{i,j}^*(\omega_i) = \omega_j$$

Exercise: Let $\Lambda \subset \mathbb{C}$ be a lattice and let $T := \mathbb{C}/\Lambda$ be a complex torus. Let z be the standard variable on $\mathbb{C}$. Show that dz is a degree 1 differential on T.

Meromorphic modular forms

Definition: Let Γ be a congruence subgroup of $SL_2(\mathbb{Z}) =: \Gamma_1$.

A meromorphic function f on $\mathfrak{h}$ is a **meromorphic modular form of weight k** if

- $f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$ for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ and $z \in \mathfrak{h}$.
- Around each cusp $\alpha = \frac{a_1\infty + b_1}{c_1\infty + d_1}$, $\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \in \Gamma_1$, f has the Laurent expansion:

$$(cz+d)^{-k} f\left(\frac{a_1 z + b_1}{c_1 z + d_1}\right) = \sum_{n=n_0}^{\infty} c_{f,\alpha}(n)\, e^{\frac{2\pi i}{h_\alpha}\cdot n z}, \quad n_0 \in \mathbb{Z}$$

for some $h_\alpha \in \mathbb{Z}_{>0}$

Notation: The space of meromorphic modular forms of weight k and group Γ is denoted $\mathcal{A}_k(\Gamma)$.

Meromorphic differentials on a modular curve

Let Γ be a congruence subgroup

Proposition: Let f be a meromorphic modular form of weight $k \in 2\mathbb{Z}_{>0}$ for congruence subgroup Γ. Then the meromorphic differential $f(z)(dz)^{k/2}$ is invariant under the action of Γ on $\mathfrak{h}$.

Proof: $\omega := f(z)(dz)^{k/2}$ $\quad \alpha : \mathfrak{h} \to \mathfrak{h}, \; z \longmapsto \frac{az+b}{cz+d}$

$$\alpha^*(\omega) = f\left(\frac{az+b}{cz+d}\right)\left(\frac{d\alpha}{dz}\right)^{k/2}(dz)^{k/2} = f(z)(cz+d)^{k} \cdot (cz+d)^{-2\cdot\frac{k}{2}} \cdot (dz)^{k/2} = \omega$$

$\blacksquare$

Meromorphic differentials on a modular curve

$$\mathfrak{h} \xrightarrow{\pi} X(\Gamma)$$

$$\begin{array}{ccc} \Omega^{\otimes k/2}(\mathfrak{h}) & \longleftarrow & \Omega^{\otimes k/2}(X(\Gamma)) \\ f(z)(dz)^{k/2} & \overset{\pi^*}{\longleftarrow\!\shortmid} & \omega \end{array}$$

Proposition: For $f \in \mathcal{A}_k(\Gamma)$ there exists a unique meromorphic differential $\omega \in \Omega^{\otimes k/2}(X(\Gamma))$ such that $\pi^*(\omega) = f(z)(dz)^{k/2}$.

Theorem: The map from $\mathcal{A}_k(\Gamma)$ to $\Omega^{\otimes k/2}(X(\Gamma))$ given by $f \longmapsto \omega$ is an isomorphism of vector spaces.

Proof of the proposition: Let $f \in A_k(\Gamma)$

We have to define ω in local coordinates

Let $P \in X(\Gamma)$. Consider the following cases:

- P is a regular point.

  $\pi^{-1} : \mathcal{U}_p \to \mathfrak{h}$ is a local coordinate

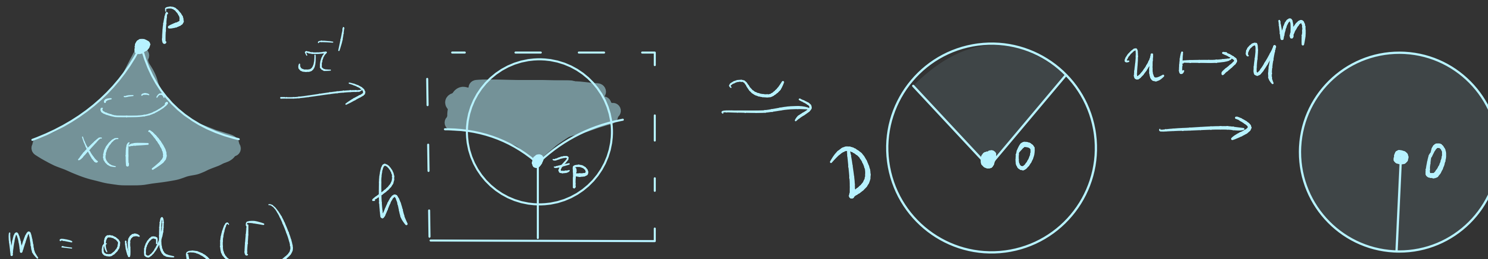
  $$\omega = (\pi^{-1})^* \left(f(z)(dz)^{k/2} \right)$$

- P is an elliptic point

- P is a cusp

Recall:
$\mathcal{D} = \{ z \mid |z| < 1 \}$

- P is an elliptic point



$m = \mathrm{ord}_P(\Gamma)$

Let A be a generator of $\mathrm{Stab}_{z_P}(\bar{\Gamma})$. There exists a biholomorphic map $u : \mathfrak{h} \to D$ such that $u(Az) = e^{2\pi i/m} \cdot u(z)$

The variable $q := u^m$ is a local coordinate around P on $X(\Gamma)$.

We have $f(z)\,(dz)^{k/2} = f(z(u)) \left(\frac{dz}{du}\right)^{k/2} (du)^{k/2} =: \tilde{f}(u)\,(du)^{k/2}$

Since u is a biholomorphic map we have

$\mathrm{ord}_{z=z_P} f(z) = \mathrm{ord}_{u=0}\, \tilde{f}(u)$. Moreover, $\tilde{f} : D \to \mathbb{P}^1$ is meromorphic

As $f(z)(dz)^{k/2}$ is invariant under the pullback by A,
then $\tilde{f}(u)(du)^{k/2}$ is invariant under $u \mapsto e^{2\pi i/m} \cdot u$

Thus $\tilde{f}(e^{2\pi i/m} \cdot u)\,(d(e^{2\pi i/m} u))^{k/2} = \tilde{f}(u)(du)^{k/2}$

and (✳) $\tilde{f}(e^{2\pi i m} \cdot u) = e^{\frac{-\pi i k}{m}} \cdot \tilde{f}(u)$

The function $\tilde{f}(u)$ has a Laurent expansion around $u=0$

(☆) $$\tilde{f}(u) = \sum_{n=n_0}^{\infty} c_{\tilde{f}}(n)\, u^n \quad , \quad n_0 \in \mathbb{Z}$$

Equation (✳) implies that (☆) is of the form

$$\tilde{f}(u) = \sum_{\substack{n \equiv -k/2 \bmod m \\ n \geq n_0}} c_{\tilde{f}}(n)\, u^n$$

Now we express the form $\tilde{f}(u)(du)^{k/2}$ in the local coordinate $q = u^m$.

The expansion $\tilde{f}(u) = \sum_{\substack{n \equiv -k/2 \bmod m \\ n \geq n_0}} c_{\tilde{f}}(n)\, u^n$

implies $\tilde{f}(u) = u^{-k/2} \cdot \tilde{\tilde{f}}(q)$, where $\tilde{\tilde{f}}$ is meromomorphic.

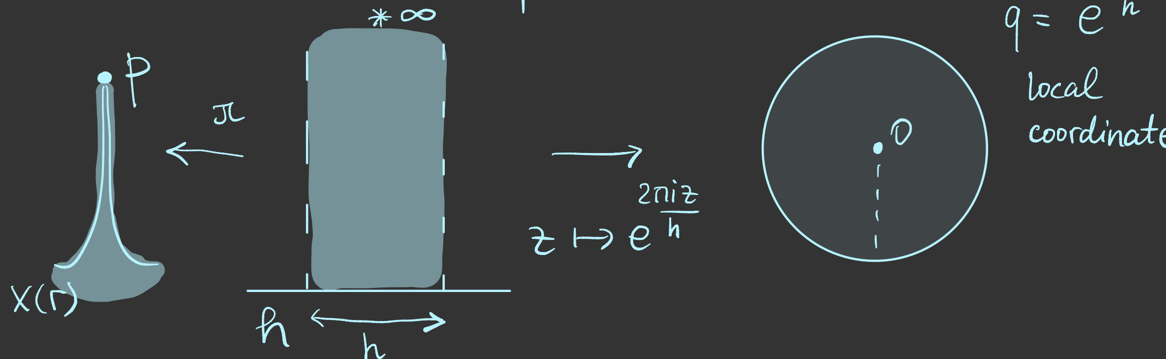
By the chain rule we have $dq = m\, u^{m-1}\, du$.

Therefore $du = \frac{1}{m}\, u\, q^{-1}\, dq$

$$\bullet + \bullet \Rightarrow (\pi^{-1})^* \left(\tilde{f}(u)\,(du)^{k/2} \right) = u^{-k/2}\, \tilde{\tilde{f}}(q)\, \frac{1}{m^{k/2}}\, u^{k/2}\, q^{-k/2}\, (dq)^{k/2}$$

$$\pi^{-1} : \underset{X(\Gamma)}{\overset{\cap}{U_P}} \rightarrow \mathfrak{h} \qquad = \frac{1}{m^{k/2}}\, \tilde{\tilde{f}}(q)\, q^{-k/2}\, (dq)^k$$

- P is a cusp



$$dq = \frac{2\pi i}{h} e^{\frac{2\pi i}{h} z} \, dz \qquad dz = \frac{h}{2\pi i} q^{-1} \, dq$$

$$(\pi^*)^{-1}\left(f(z) \, (dz)^{k/2} \right) = f(z(q)) \left(\frac{h}{2\pi i}\right)^{k/2} q^{-k/2} \, (dq)^{k/2}$$

meromorphic function in a local chart

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More details in „The first course on modular forms" Section 3.3

Divisors on Riemann surfaces

Definition: Let X be a Riemann surface.
A **divisor** on X is a finite formal sum of integer multiples of points on X

$$D = \sum_{x \in X} n_x \cdot x \qquad \begin{array}{l} n_x \in \mathbb{Z} \text{ for all } x, \\ n_x = 0 \text{ for almost all } x. \end{array}$$

The set $\mathrm{Div}(X)$ is the free Abelian group on the points of X.

$$\sum_{x \in X} m_x \cdot x + \sum_{x \in X} n_x \cdot x = \sum_{x \in X} (m_x + n_x) x$$

The **degree** of a divisor D is $\deg(D) = \sum n_x$

Divisor of a meromorphic function

Let f be a non-zero meromorphic function on X

For a point $x \in X$

$$\mathrm{ord}_f(x) := \begin{cases} \mathrm{mult}_f(x) & \text{if } f(x) = 0 \\ -\mathrm{mult}_f(x) & \text{if } f(x) = \infty \\ 0 & \text{otherwise} \end{cases}$$

Definition: The **divisor** of f is defined as

$$\mathrm{div}(f) := \sum_{x \in X} \mathrm{ord}_f(x) \cdot x$$

Lemma: The map $\mathrm{div}: \mathbb{C}(X)^* \to \mathrm{Div}(X)$ is a homomorphism:

$$\mathrm{div}(f_1 \cdot f_2) = \mathrm{div}(f_1) + \mathrm{div}(f_2).$$

Lemma: For every $f \in \mathbb{C}(X)^*$ the divisor $\mathrm{div}(f)$ has degree 0.

Proof: exercise.

Terminology: $\mathrm{div}(f)$ is a **principal** divisor.

Divisor of a meromorphic differential

Let ω be a meromorphic differential of degree k on a Riemann surface X.

Let $x \in X$ be a point and let z be a local coordinate around x s.t $z(x) = 0$.

Then ω has the form

$\omega = f(z)(dz)^k$, where f is a meromorphic function

f has the Laurent expansion around $z=0$

$$f(z) = \sum_{n \geq n_0} a_n \cdot z^n \quad , \quad n_0 \in \mathbb{Z}$$

Definition: $\operatorname{ord}_x \omega = n_0$

Lemma: The number n_0 does not depend on a choise of local coordinate

Divisor of a meromorphic differential

Definition: Let ω be a meromorphic differential

$$\operatorname{div}(\omega) := \sum_{x \in X} \operatorname{ord}_x(\omega) \cdot x$$

Definition: A **canonical divisor** on X is a divisor of the form $\operatorname{div}(\lambda)$ where λ is a nonzero element of $\Omega^1(X)$

Exercise. Let $\lambda, \mu \in \operatorname{Div}(X)$. Suppose that λ is a canonical divisor and μ is a principal divisor. Then $\lambda + \mu$ is also a canonical divisor

Notation: Let $D = \sum_x n_x \cdot x$ be a divisor on X

we write $D \geq 0$ if $n_x \geq 0$ for all $x \in X$.

Let D be any divisor on X. Consider the following vector space:

$$L(D) = \{ f \in \mathbb{C}(X) \mid f \equiv 0 \text{ or } \operatorname{div}(f) + D \geq 0 \}$$

$$\ell(D) := \dim L(D)$$

Theorem (Riemann-Roch) Let X be a compact Riemann surface of genus g. Let $\operatorname{div}(\lambda)$ be a canonical divisor on X. Then for any divisor $D \in \operatorname{Div}(X)$

$$\ell(D) = \deg(D) - g + 1 + \ell(\operatorname{div}(\lambda) - D)$$

Corollary: Let X be a compact Riemann surface of genus g.
Let $\operatorname{div}(\lambda)$ be a canonical divisor on X.
Let D be any divisor on X

(a) $\ell(\operatorname{div}(\lambda)) = g$

(b) $\deg(\operatorname{div}(\lambda)) = 2g-2$

(c) If $\deg(D) < 0$ then $\ell(D) = 0$

(d) If $\deg(D) > 2g-2$ then $\ell(D) = \deg(D) - g + 1$

Proof: (c) Suppose that $f \in L(D)$

Then $\operatorname{div}(f) + D \geq 0 \implies \deg(\operatorname{div}(f) + D) \geq 0$

On the other hand: $\deg(\operatorname{div}(f)) = 0$, $\deg(D) < 0$

Hence $\deg(\operatorname{div}(f) + D) < 0$ ↯

Definition: A differential ω of degree n on a Riemann surface X is **holomorphic** if at each point $x \in X$ there exists a local coordinate z such that $\omega = f(z)(dz)^n$ where f is holomorphic.

Definition: A modular form $f \in \mathcal{M}_k(\Gamma)$ is a **cusp form** if for each cusp $\alpha = \frac{a\infty + b}{c\infty + d}$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1$ f has the expansion

$$(cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right) = \sum_{n=1}^{\infty} c_{f,\alpha}(n)\, e^{\frac{2\pi i z}{h_\alpha}}$$

$S_k(\Gamma)$ is the space of **cusp forms** of weight k and group Γ.

Lemma: $\Omega^1_{hol}(X(\Gamma)) \simeq S_2(\Gamma)$

Exercise: Show that $\dim S_2(X(\Gamma)) = g(X(\Gamma))$

Proof of the lemma:

$$\mathfrak{h} \xrightarrow{\pi} X(\Gamma)$$

Suppose that $\omega \in \Omega^1_{hol}(X(\Gamma))$.

By the theorem on p. 17

$$\pi^*(\omega) = f(z)\,dz \quad \text{for some } f \in \mathcal{A}_2(\Gamma).$$

Let $x \in X(\Gamma)$ be a regular point

$x = \pi(z_x)$, $z_x \in \mathfrak{h}$, $\mathrm{Stab}(z_x, \overline{\Gamma}) = \{\overline{id}\}$.

$$\boxed{\mathrm{ord}_x(\omega) = \mathrm{ord}_{z=z_x}(f(z)\,dz).}$$

f is holomorphic at z_x if and only if ω if holomorphic at x.

Let $y \in X(\Gamma)$ be an elliptic point of order m.

$y = \pi(z_y)$, $z_y \in \mathfrak{h}$, $|\mathrm{Stab}(z_y, \overline{\Gamma})| = m$.

$y = \pi(z_y)$, $z_y \in \mathfrak{h}$ $\quad q = (z - z_y)^m \quad dq = m(z-z_1)^{m-1} dz$

$$\omega = (\pi^*)^{-1}(f(z)\,dz) = f(z(q))\, m^{-1} (z - z_y)^{1-m}\, dq$$

$$= m^{-1} f(z(q))\, q^{\frac{1}{m}-1}\, dq$$

Laurent expansions near z_y and y:

$$f(z)\,dz = \sum_{n \equiv -1 \bmod m} c^n \cdot (z - z_y)^n\, dz$$

$$\omega(q) = m^{-1} \sum_{n \equiv -1 \bmod m} c^n\, q^{\frac{n+1}{m}-1}\, dq.$$

If

$$\operatorname{ord}_{z=z_y}(f(z)\,dz) = n_0 \quad , \quad n_0 = -1 \bmod m,$$

then

$$\operatorname{ord}_y(\omega) = \frac{n_0+1-m}{m}. \quad \text{Thus}$$

$$\boxed{\operatorname{ord}_y(\omega) = \frac{1-m}{m} + \frac{1}{m}\cdot \operatorname{ord}_{z_y}(f(z)\,dz).}$$

We observe, that $\frac{n_0}{m} - \frac{m-1}{m} \geq 0$ if and only if $n_0 \geq 0$.

Thus ω is holomorphic at point y if and only if $f(z)$ is holomorphic at $z = z_y$.

Let $\infty \in X(\Gamma)$ be cusp of width h.

$$\mathrm{Stab}(\infty, \bar{\Gamma}) = \left\{ \begin{pmatrix} 1 & h\cdot n \\ 0 & 1 \end{pmatrix} \,\middle|\, n \in \mathbb{Z} \right\}.$$

$$q = e^{\frac{2\pi i}{h}\cdot z} \qquad dq = \frac{2\pi i}{h}\cdot q \cdot dz$$

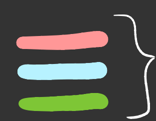
$$\omega = \pi^*\left(f(z)\,dz\right) = \frac{h}{2\pi i} f(z(q))\, q^{-1}\, dq$$

Laurent expension at ∞:

$$f(z)\,dz = \sum_{n=n_0}^{\infty} c_n e^{\frac{2\pi i}{h}\cdot nz}\, dz, \Rightarrow \boxed{\mathrm{ord}_\infty(\omega) = n_0 - 1}.$$

$$\omega = \sum_{n=n_0}^{\infty} c_n q^{n-1}\, dq$$

Modular form f vanishes at ∞ if and only if ω is holomorphic at ∞.


$\Rightarrow \blacksquare$.